# Some Properties of Image Circles

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**Summary**—Some properties of image circles for four-terminal networks are discussed. A procedure for correcting an image circle, obtained with a lossy short, is presented. Indirect methods for determining the open and short circuit impedances of a symmetrical four-terminal network are discussed.

## The Image Circle

The IMAGE circle for a linear and bilateral four-terminal network, or two-port waveguide structure, is obtained in the following manner:

a) A variable reactance, or a movable short, is connected to the output terminals, or port. It is assumed that the reactance, or short, is lossless.

b) The input impedances, or voltage reflection coefficients, for several values of reactance, or positions of the short, are measured.

c) The input impedances, or reflection coefficients, are plotted on a circular transmission line chart; *i.e.*, Smith $(R-X)$ or Carter $(Z-\theta)$ chart, or polar graph paper.[1] Any convenient normalizing impedance can be used with the transmission line chart.

d) The image circle is drawn through these points. If all of the points do not lie on a circle, this is due to experimental errors. Hereafter, it is assumed that the image circle is drawn on a Smith chart.

The image circle has several interesting properties which were discussed in many papers and research reports. Here two experimental procedures are presented making use of the image circle. Theoretical basis for these is discussed in Appendix. Before presenting these procedures, it is desirable to consider the input impedance when network is terminated in a resistance.

## Input Impedance for Resistive Load

The input impedance $\boldsymbol{Z}_A$, when the network is terminated in a resistance $R_2$, can often be determined by direct measurement. If $R_2$ is the characteristic impedance of the transmission line or waveguide containing the movable short, the input impedance can be determined by indirect methods which make use of the image circle.[2–5] These methods do not require that the image circle be obtained with a lossless short.

These indirect methods pair points on the image circle corresponding to positions of the movable short which are separated by one-quarter wavelength. The *cross-over point*, which is used later, is defined as the intersection of the straight lines connecting these pairs of points. If the network is terminated in a variable reactance, these indirect methods can be used if the points on the image circle corresponding to the reactances $X$ and $-R_2^2/X$ are paired.

## A Procedure for Correcting an Image Circle Obtained with a Lossy Short

The magnitude $\Gamma$ of the voltage reflection coefficient of the short is determined. The points for the image circle are plotted. The center $\boldsymbol{Z}_B$ of the image circle is located and the image circle is drawn, as shown in Fig. 1. The value of the input impedance $\boldsymbol{Z}_A$ for $R_2$ equal to the characteristic impedance of the line containing the movable short is determined and plotted.

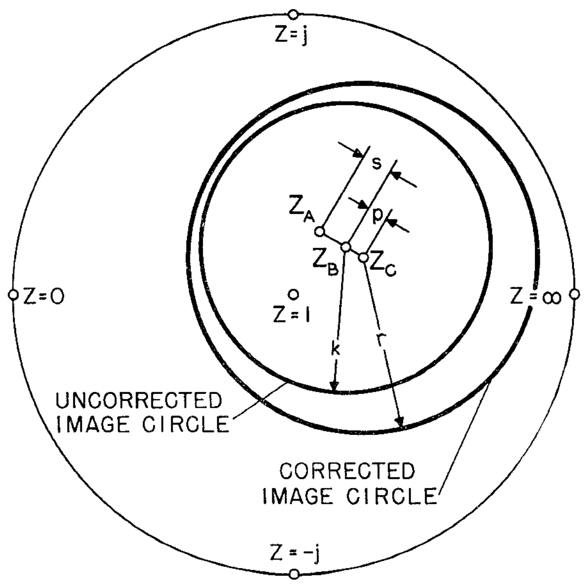


Fig. 1—A circular impedance diagram that illustrates the procedure for correcting an image circle.

The radius $k$ of the image circle and the distance $s$ between the points $\boldsymbol{Z}_A$ and $\boldsymbol{Z}_B$ are measured. The value of

$$p = \frac{(1-\Gamma^2)k^2 s}{\Gamma^2 k^2 - s^2} \tag{1}$$

is calculated. The center $\boldsymbol{Z}_C$ of the corrected image circle is located on the straight line through $\boldsymbol{Z}_A$ and $\boldsymbol{Z}_B$, as shown in Fig. 1. The point $\boldsymbol{Z}_B$ always lies between $\boldsymbol{Z}_A$ and $\boldsymbol{Z}_C$. The radius $r$ of the corrected image circle is given by

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[1] The reader should recall that plotting impedances on a circular transmission line chart is equivalent to plotting voltage reflection coefficients.

[2] G. A. Deschamps, "A new chart for the solution of transmission line and polarization problems," Trans. IRE, vol. MTT-1, pp. 5–13; March, 1953; also, *Electrical Communications*, vol. 30, pp. 247–54; September, 1953.

[3] G. A. Deschamps, "Determination of reflection coefficients and insertion loss of a waveguide junction," *Jour. Appl. Phys.*, vol. 24, pp. 1046–50; August, 1953; also, *Electrical Communications*, vol. 31, pp. 57–62; March, 1954.

[4] G. A. Deschamps, "A Hyperbolic Protractor for Microwave Impedance Measurements and Other Purposes," Federal Telecommunication Laboratories, Nutley, N. J.; 1953.

[5] J. E. Storer, L. S. Sheingolb, and S. Stein, "A simple graphical analysis of a two-port waveguide junction," Proc. IRE, vol. 41, pp. 1004–13; August, 1953.

$$r = \frac{(k^2 - s^2)\Gamma k}{\Gamma^2 k^2 - s^2}. \tag{2}$$

Finally, the corrected image circle is drawn using the center $Z_C$ and the radius $r$.

### Indirect Procedures for Determining $Z_{sc}$ and $Z_{oc}$ of a Symmetrical Four-Terminal Network

For some convenient value of $R_2$, the value of $Z_A$ is determined. The points for the image circle and the point $Z_A$ are plotted, using $R_2$ as the normalizing impedance. The center $Z_C$ of the image circle is located and the image circle is drawn, as shown in Fig. 2.

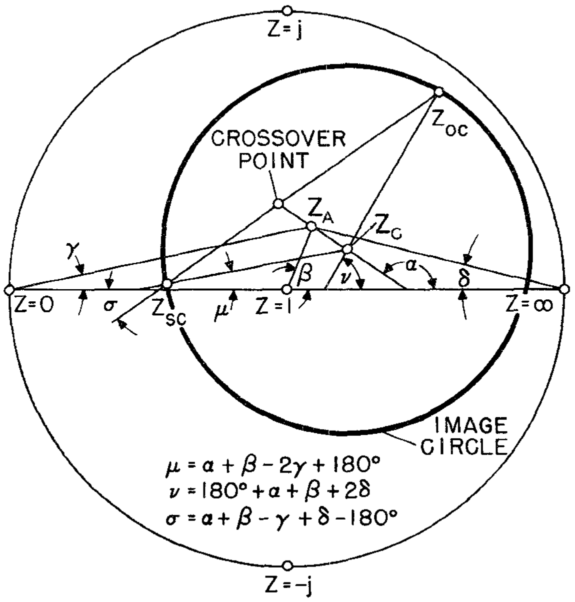


Fig. 2—A circular impedance diagram that illustrates the graphical procedure for determining $Z_{sc}$ and $Z_{oc}$.

Straight lines are drawn through the points $Z_A$ and the points $Z_C$, $Z=0$, 1, and $\infty$. The angles $\alpha$, $\beta$, $\gamma$, and $\delta$ are measured. Lines are drawn through the points $Z_C$ which makes the angles

$$\mu = \alpha + \beta - 2\gamma - 180° \tag{3}$$

and

$$\nu = \alpha + \beta + 2\delta + 180° \tag{4}$$

with the $X=0$ line. The intersections of these lines with the image circle are $Z_{sc}$ and $Z_{oc}$, respectively.

If the crossover point is known, the line through this point which makes the angle

$$\sigma = \alpha + \beta - \gamma + \delta - 180° \tag{5}$$

with the $X=0$ line intersects the image circle at $Z_{sc}$ and $Z_{oc}$. When the crossover point is above the $X=0$ line, the point $Z_{sc}$ is nearer the $X=0$ line if $0<\sigma<180°$ and $Z_{oc}$ is nearer if $180°<\sigma<360°$. When the crossover point is below the $X=0$ line, $Z_{oc}$ is nearer the $X=0$ line if $\sigma<\sigma<180°$ and $Z_{sc}$ is nearer if $180°<\sigma<360°$. If $0=0$, $Z_{sc}$ is left of $Z_{oc}$; and if $\sigma=180°$, $Z_{sc}$ is right of $Z_{oc}$.

The triangles whose vertices are $Z=0$, $Z_{oc}$, $1/Z_{sc}$, and $Z=0$, $Z_A$, $1/Z_A$ should be similar. These triangles can be drawn and used to check the results.

The values of $Z_{sc}$ and $Z_{oc}$ can also be determined from the values of $Z_A$ and $Z_C$ by using

$$Z_{sc} = \frac{(Z_A + \bar{Z}_A)Z_C - Z_A\bar{Z}_A(1 + Z_C)}{Z_A - \bar{Z}_A Z_C} \tag{6}$$

and

$$Z_{oc} = \frac{Z_A - \bar{Z}_A Z_C}{1 - Z_A - \bar{Z}_A + Z_C}, \tag{7}$$

where $\bar{Z}_A$ is the conjugate of $Z_A$. Let $K_A$, $\bar{K}_A$, and $K_C$ denote the voltage reflection coefficients corresponding to $Z_A$, $\bar{Z}_A$, and $Z_C$, respectively. Eqs. (6) and (7) are equivalent to

$$Z_{sc} = -\frac{K_A + \bar{K}_A - K_C + K_A\bar{K}_A(K_C + 2)}{K_A - \bar{K}_A - (1 - K_A\bar{K}_A)K_C}, \tag{8}$$

$$Z_{oc} = -\frac{K_A - \bar{K}_A - (1 - K_A\bar{K}_A)\bar{K}_C}{K_A + \bar{K}_A - K_C + K_A\bar{K}_A(K_C - 2)}, \tag{9}$$

$$K_{sc} = \frac{K_A - K_C + K_A\bar{K}_A(K_C + 1)}{\bar{K}_A + K_A\bar{K}_A}, \tag{10}$$

and

$$K_{oc} = -\frac{K_A - K_C + K_A\bar{K}_A(K_C - 1)}{\bar{K}_A - K_A\bar{K}_A}. \tag{11}$$

The writer has found that the graphical procedure which was described first gives the most accurate results of any of the procedures presented in this paper for determining $Z_{sc}$ and $Z_{oc}$.

## Appendix

### *General Theory*

According to well-known circuit theory, the input impedance $Z_1$ and the load impedance $Z_2$ of a linear and bilateral four-terminal network are related by

$$Z_1 = \frac{AZ_2 + B}{CZ_2 + D}. \tag{12}$$

Let $K_1$ denote the voltage reflection coefficient at the input terminals relative to the resistance $R_1$; *i.e.*

$$K_1 = \frac{Z_1 - R_1}{Z_1 + R_1}; \tag{13}$$

and let $K_2$ denote the voltage reflection coefficient at the output terminals relative to the resistance $R_2$; *i.e.*

$$K_2 = \frac{Z_2 - R_2}{Z_2 + R_2}. \tag{14}$$

Eqs. (12)–(14) can be combined to obtain

$$K_1 = \frac{aK_2 + b}{cK_2 + d}. \tag{15}$$

Eqs. (12) and (15) are called bilinear transformations. Bilinear transformations map circles into circles.

Eq. (15) can be written

$$K_1 = \frac{a\bar{c}K_2\bar{K}_2 - b\bar{d}}{c\bar{c}K_2\bar{K}_2 - d\bar{d}} - \frac{(\overline{cK_2 + d})(ad - bc)K_2}{(cK_2 + d)(c\bar{c}K_2\bar{K}_2 - d\bar{d})}, \quad (16)$$

where the bars above the symbols denote conjugates. If $K_2$ lies on a circle whose center is at the origin and whose radius is $\Gamma$, it follows from (16) that $K_1$ lies on a circle whose center $K_B$ and radius $k$ are given by

$$K_B = \frac{a\bar{c}\Gamma^2 - b\bar{d}}{c\bar{c}\Gamma^2 - d\bar{d}} \quad (17)$$

and

$$k = \left| \frac{ad - bc}{c\bar{c}\Gamma^2 - d\bar{d}} \right| \Gamma. \quad (18)$$

If $Z_2 = jX$, where $X$ is a variable reactance; $K_2$ lies on the circle defined by $\Gamma = 1$ and $K_1$ lies on the circle which is called the image circle. The center and radius of the image circle are denoted by $K_C$ and $r$, respectively. When $Z_2 = R_2$, $K_2 = 0$, and (15) reduces to

$$K_A = b/d. \quad (19)$$

If the image circle is obtained with a lossy short, $\Gamma < 1$. In this case, (17)–(19) can be solved for $a\bar{c}$, $b\bar{d}$, $c\bar{c}$, and $d\bar{d}$ to obtain

$$a\bar{c} = \frac{\Delta[k^2 K_A - (k^2 - s^2)K_B]}{\Gamma k(k^2 - s^2)}, \quad (20)$$

$$b\bar{d} = \frac{\Delta \Gamma k\, K_A}{k^2 - s^2}, \quad (21)$$

$$c\bar{c} = \frac{\Delta s^2}{\Gamma k(k^2 - s^2)}, \quad (22)$$

and

$$d\bar{d} = \frac{\Delta \Gamma k}{k^2 - s^2}; \quad (23)$$

where

$$|\Delta| = |ad - bc| \quad (24)$$

and

$$s = |K_A - K_B|. \quad (25)$$

The center of the corrected image circle is given by

$$K_C = \frac{a\bar{c} - b\bar{d}}{c\bar{c} - d\bar{d}} = K_B + (p/s)(K_B - K_A), \quad (26)$$

where $p$ is given by (1). The radius $r$ is given by

$$r = \left| \frac{\Delta}{c\bar{c} - d\bar{d}} \right| = \frac{(k^2 - s^2)\Gamma k}{\Gamma^2 k^2 - s^2}. \quad (2')$$

Thus (1) and (2) can be used to correct an image circle which was obtained with a lossy short.

In the following discussion, it is assumed that the four-terminal network is symmetrical. Now, $A = D$ and $b = -c$. The center $K_C$ of the image is given by

$$K_C = -\frac{a\bar{b} + b\bar{d}}{b\bar{b} - d\bar{d}}. \quad (27)$$

Eqs. (19) and (27) can be solved for $a/d$ to obtain

$$\frac{a}{d} = -\frac{K_A - (1 - K_A\bar{K}_A)K_C}{\bar{K}_A}. \quad (28)$$

If $Z_2 = 0$, $K_2 = -1$ and (15) can be written

$$K_{sc} = -\frac{\dfrac{a}{d} - \dfrac{b}{d}}{\dfrac{b}{d} + 1}. \quad (29)$$

If the values of $a/d$ and $b/d$ given by (28) and (19) are substituted in (29), the result is (10). When $Z_2 = \infty$, $K_2 = 1$ and (15) can be written

$$K_{oc} = -\frac{\dfrac{a}{d} + \dfrac{b}{d}}{\dfrac{b}{d} - 1}. \quad (30)$$

If the values of $a/d$ and $b/d$ given by (28) and (19) are substituted in (30), the result is (11). Eqs. (6)–(9) can be obtained from (10) and (11).

It follows from (10) and (11) that

$$K_{sc} - K_C = \frac{(K_A - K_C)(1 + \bar{K}_A)}{\bar{K}_A(1 + K_A)}, \quad (31)$$

$$K_{oc} - K_C = -\frac{(K_A - K_C)(1 - \bar{K}_A)}{\bar{K}_A(1 - K_A)}, \quad (32)$$

and

$$K_{sc} - K_{oc} = 2\frac{(K_A - K_C)(1 - K_A\bar{K}_A)}{\bar{K}_A(1 + K_A)(1 - K_A)}. \quad (33)$$

The angles of the vectors in (31)–(33) must satisfy

$$\angle(K_{sc} - K_C) = \angle(K_A - K_C) + \angle K_A - 2\angle(1 + K_A), \quad (34)$$

$$\angle(K_{oc} - K_C) = 180° + \angle(K_A - K_C) + \angle K_A - 2\angle(1 - K_A), \quad (35)$$

and

$$\angle(K_{sc} - K_{oc}) = \angle(K_A - K_C) + \angle K_A - \angle(1 + K_A) - \angle(1 - K_A). \quad (36)$$

Eqs. (34)–(36) are equivalent to (3)–(5).

It follows from (10) and (11) that

$$\frac{1 + K_{oc}}{1 - K_{sc}} = \frac{1 + K_A}{1 - K_A}. \quad (37)$$

This equation requires that the triangles $(-1, K_{oc}, -K_{sc})$ and $(-1, K_A, -K_A)$ be similar. This is equivalent to requiring that the triangles whose vertices are $Z = 0$, $Z_{oc}$ $1/Z_{sc}$ and $Z = 0$, $Z_A$, $1/Z_A$ be similar.